Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Graph Theory

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Max. points : 50. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. If you are citing results from assigments, you need to prove them. Answer any five questions. Only the first five answers will be evaluated.

1. (a) Show that the line graph of a connected graph is connected. (**5**)
   (b) Let G be a connected graph which is isomorphic to its line graph. Show that G is a cycle graph. (**5**)
2. Let e be an edge in K_n, the complete graph on n-vertices. Show that the number of labelled spanning trees in $K_n - e$ is $(n-2)n^{n-3}$. (**10**)
3. A 3-regular simple graph G has a 1-factor iff it decomposes (i.e., edge-disjoint union) into copies of P_4, the path graph on 4 vertices. (**10**)
4. Characterize the graphs G for which the following statements hold. Justify your answers. (**10**)
   (1) (max. independent set) $\alpha(G) = 1$.
   (2) (max. size of matching) $\alpha'(G) = 1$.
   (3) (min. vertex cover) $\beta(G) = 1$.
   (4) (min. edge cover) $\beta'(G) = 1$.
   NOTE : In each of the above, you are required to prove a statement of the form $\ldots(G) = 1$ *iff* G *is* $\ldots$.
5. Let G be a 2-edge connected graph. Define a relation R on $E(G)$ by $(e, f) \in R$ if $e = f$ or if $G - e - f$ is disconnected. Show the following.

   (1) $(e, f) \in R$ iff e, f belong to the same set of cycles (i.e., for a cycle C, $e \in C$ iff $f \in C$.) (**4**)

(2) Show that R is an equivalence relation. (**3**).
(3) Let $[e]$ denote the equivalence class of e and let C be a cycle. Show that $e \in C$ iff $[e] \subset C$. (**3**)

6. (a) A plane graph is a k-angulation if every face has length k. Let G be a connected graph and a k-angulation on n vertices and m edges. Show that $m = (n-2)\frac{k}{k-2}$. (**3**)

   (b) Is there a K_5 minor of the Petersen graph ? (**2**)

   (c) Show that the Petersen graph has a $K_{3,3}$ minor. (Hint : Start by deleting one vertex and do not delete any edges.) (**5**).

7. Show that the adjacency matrix of a tree is totally unimodular i.e., every square submatrix has determinant $0, -1, +1$. Does the statement hold for adjacency matrices of arbitrary graphs as well ?(**10**).

8. Let G be a graph on $[n]$ with connected components $C_1, \ldots C_k$ for some $k \geq 1$. For $i = 1, \ldots, k$ define the vectors $x^i := (x^i_j)_{j=1,\ldots,n} \in \mathbb{R}^n$ as follows : $x^i_j = 1[j \in C_i]$. Show that the vectors $x^i, i = 1, \ldots, k$ form a basis for $Ker(\mathcal{L}_G)$ where $\mathcal{L}_G : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation defined by $\mathcal{L}_G(x) := L_G x^T$ with L_G denoting the Laplacian matrix of the graph G. (**10**)